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Special Inclinations Allowing Minimal Drift Orbits for Formation Flying Satellites

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The possibility of obtaining a natural periodic relative motion of formation flying Earth satellites is investigated both numerically and analytically. The numerical algorithm is based on a genetic strategy, refined by means of nonlinear programming, that rewards periodic relative trajectories. First, we test our algorithm using a point mass gravitational model. In this case the period matching between the considered orbits is a necessary and sufficient condition to obtain invariant relative trajectories. Then, the J_2 perturbed case is considered. For this case, the conditions to obtain an invariant relative motion are known only in approximated closed forms which guarantee a minimal orbit drift, not a motion periodicity. Using the proposed numerical approach, we improved those results and found two couples of inclinations (63.4 and 116.6 deg, the critical inclinations, and 49 and 131 deg, two new "special" inclinations) that seemed to be favored by the dynamic system for obtaining periodic relative motion at small eccentricities.

Nomenclature

a	=	orbit semimajor axis
a_i, b_i	=	epicyclic orbital elements
e	=	orbit eccentricity
$\mathbf{f}$	=	bector function representing the satellites relative dynamics
f_r, f_θ, f_z	=	perturbing forces in r, θ, z coordinate system
i	=	orbit inclination
J	=	objective function for the genetic optimizer
J_2	=	Earth flattening term in gravitational field series expansion
M	=	orbit mean motion
N_{gen}	=	number of generations in the genetic optimizer
N_{ind}	=	number of individuals in the genetic optimizer
$\boldsymbol{P}$	=	perturbing actions
p	=	orbit semilatus rectum
$\boldsymbol{R}$	=	rotation matrix from inertial to LVLH frame
$R_\oplus$	=	Earth equatorial radius
$\boldsymbol{r} = [X \;\; Y \;\; Z]$	=	satellite absolute position vector in inertial frame
T	=	candidate period
t	=	time
$\mathbf{x} = [x \;\; y \;\; z \;\; \dot{x} \;\; \dot{y} \;\; \dot{z}]$	=	satellites relative state vector in Cartesian coordinates
α_i, β_i	=	canonical momenta
α_0	=	satellites initial relative phase angle
ΔV	=	velocity instantaneous variation
ε	=	perturbation term
θ	=	on orbit satellite anomaly
$\boldsymbol{\kappa}$	=	decision vector for the genetic optimizer
μ	=	Earth gravitational constant
ϖ	=	orbit argument of perigee
ρ	=	satellites relative distance
$\boldsymbol{\rho} = [x \;\; y \;\; z]$	=	satellites relative position vector
Ω	=	orbit right ascension of ascending node
$\boldsymbol{\omega}$	=	LVLH frame angular velocity vector
$(\cdot)_0$, $(\cdot)_f$	=	value at initial and final time

I. Introduction

LATELY, numerous missions involving satellites flying in formation have been planned or studied. A brief review includes ESA missions Proba 3, LISA, XEUS, Darwin, and SMART-3, NASA's missions EO-1, ST5, and Terrestrial Planet Finder, the international mission known as A-Train and the JAXA's SCOPE mission. To keep the satellites of the formation in the designed configuration, and therefore to achieve the mission's goals, control actions are needed. The cost of this orbital control in terms of ΔV limits both the mission duration and the expected performances. Advantageous dynamics could reduce the cost of these operations, in particular, the possibility to obtain periodic or quasi-periodic natural relative motion would be a significant saving factor as recently argued by Becerra et al. [1]. Many different approaches to find a periodical relative motion are considered in the recent literature on this topic. Inalhan et al. [2] found the analytical expression for the initial conditions resulting in periodic motion based on the classical Tschauner–Hempel equations [3]. Kasdin and Koleman [4] used the epicyclic orbital elements theory to derive bounded, periodic orbits in the presence of various perturbations. Vaddi et al. [5] studied the Hill–Clohessy–Wiltshire [6] (HCW) modified system to include

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second-order terms. Finally, Schaub and Alfriend [7] formulated the conditions for invariant J_2 relative orbits introducing relations between the mean orbital elements of the two satellites. The analytical approaches taken in these works lead to two kinds of findings: 1) initial conditions that ensure exact periodicity in approximated dynamic models or 2) initial conditions resulting in bounded (i.e., with minimum drift, but not periodic) relative motion in more detailed dynamic models. On the other hand, the numerical approach taken here is able to reveal previously unknown features of the minimum drift relative trajectories for satellites in a fully nonlinear, perturbed environment.

The paper is structured as follows. In Sec. II, we define our problem as an optimization problem and we describe the numerical technique we use to solve it. In Sec. III, we test the behavior of the algorithm by applying it to the well-known and simple unperturbed relative satellite motion case. In Sec. IV we introduce the J_2 perturbation and we apply our algorithm to this case commenting on the unexpected results.

II. Problem Statement

Consider a generic nonautonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, for example, the relative dynamics of satellites flying in formation. Define $\boldsymbol{\delta}\mathbf{x} = \mathbf{x}_0 - \mathbf{x}(T)$, where $\mathbf{x}_0$ is the system state at the initial time and T is a time variable here called "candidate period" for reasons that will soon be clear. Then, the following optimization problem is defined:

$$\begin{cases} \text{find} & [\mathbf{X_0} \;\; T]^T \\ \text{to maximize} & J = J(|\boldsymbol{\delta}\mathbf{x}|) \\ \text{subject to} & \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \end{cases} \tag{1}$$

where the objective function J is constructed in such a way as to have its global maximum at $\boldsymbol{\delta}\mathbf{x} = \mathbf{0}$. The optimization problem above is equivalent to the task of finding as-periodic-as-possible solutions to the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$. Exploiting this approach, we solve the problem in Eq. (1) for a number of different dynamic models representing the relative motion between satellites under different orbital environments. The as-periodic-as-possible solutions correspond, in our case, to minimal relative orbit drift. As we study a number of systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, we face different optimization complexities and objective function properties. Thinking about the relative motion between satellites moving on Keplerian orbits, the problem defined by Eq. (1) has an infinite number of solutions, corresponding to orbits with equal semimajor axis. A similar structure is also expected when the Keplerian dynamic is perturbed. As a consequence, a genetic approach, avoiding issues related to domain knowledge and being able to cope with multiple local and global minima, has been selected to perform a search in the solution space. The PIKAIA freely available software [8] was used in this work as a genetic optimizer. PIKAIA encodes the decision vector $\boldsymbol{\kappa}$ using a decimal alphabet. Table 1 shows the fundamental parameters of a genetic algorithm (GA) used in all the simulations.

The best solution returned by the genetic algorithm is then refined locally by means of a nonlinear programming numerical solver.

In our simulations the decision vector $\boldsymbol{\kappa}$ contains the initial relative position, the initial relative velocity, and the candidate period T. We consider the relative motion between two satellites: a chief and a deputy to use a popular terminology connected to formation flying research. The absolute dynamics of both the chief and the deputy are simulated propagating the inertial coordinates of the spacecraft in time,

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{P}$$

where $\boldsymbol{P}$ is the perturbing action considered, μ is the planetary constant, and $\boldsymbol{r}$ is the orbital radius vector. The relative state is then evaluated by means of Eq. (2):

$$\begin{cases} [x\ y\ z]^T = \boldsymbol{R}([X_d\ Y_d\ Z_d]^T - [X_c\ Y_c\ Z_c]^T) \\ [\dot{x}\ \dot{y}\ \dot{z}]^T = \boldsymbol{R}([\dot{X}_d\ \dot{Y}_d\ \dot{Z}_d]^T - [\dot{X}_c\ \dot{Y}_c\ \dot{Z}_c]^T) - \boldsymbol{\omega} \times [x\ y\ z]^T \end{cases} \tag{2}$$

where $\boldsymbol{R}$ is the rotation matrix from the inertial coordinate system to the local-vertical/local-horizontal (LVLH) frame in which the relative state is defined. The subscripts c and d stand for chief and deputy satellites. The orbit of the chief is considered known and the initial conditions to propagate the deputy motion are obtained transforming the relative $\boldsymbol{\delta}\mathbf{x}_0$ position into absolute coordinates inverting Eq. (2).

As PIKAIA requires the decision vector components to satisfy the constraints $\bar{\kappa}_k \in [0, 1]$, we define a simple transformation between $\boldsymbol{\kappa}$ and a new decision vector $\bar{\boldsymbol{\kappa}}$ that can be used by the optimizer. For the initial relative distances:

$$\begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} = (\begin{bmatrix} -1 & -1 & -1 \end{bmatrix} + 2\begin{bmatrix} \bar{\kappa}_1 & \bar{\kappa}_2 & \bar{\kappa}_3 \end{bmatrix})K \tag{3}$$

This limits the range of variation for the initial relative position to $[-K, K]$ km. The parameter K allows bounding the dimension of the minimum drift orbit one is interested in finding. Similarly, for the relative initial velocities we set

$$\begin{bmatrix} \dot{x}_0 & \dot{y}_0 & \dot{z}_0 \end{bmatrix} = 10^{-2}(\begin{bmatrix} -1 & -1 & -1 \end{bmatrix} + 2\begin{bmatrix} \bar{\kappa}_4 & \bar{\kappa}_5 & \bar{\kappa}_6 \end{bmatrix})K \tag{4}$$

This limits the initial velocities in the range $[-10^{-2}, 10^{-2}]$ K.

We then defined $T = T_{\text{kep}} \pm \bar{\kappa}_7 k$, where k is a properly chosen constant (some tens of seconds) and T_{kep} is the orbital period $2\pi\sqrt{a^3/\mu}$ of the chief orbit.

T is clearly a crucial parameter. At T, the final relative coordinates are compared to the initial relative coordinates, thus determining the quality of the individual. A good individual has a small $\boldsymbol{\delta}\mathbf{x}$ and its position in the individual ranking is high; therefore it has a larger chance to mate and to generate "good" sons. Its genes will survive in the next generation, and if they will be ranked first in the last generation, they will be further refined by a local optimizer and represent the set of initial conditions that generate the minimum drift relative orbit. The fitness function we used to rank the individuals is

Table 1 Parameters used for the genetic optimizer

N_{ind} = no. of individuals	20
N_{gen} = no. of generations	500
No. of significant digits (no. of genes)	9
Crossover probability	0.85
Mutation mode	One-point, adjustable rate based on fitness
Initial mutation rate	0.005
Minimum mutation rate	0.0005
Maximum mutation rate	1
Reproduction plan	Steady state/replace worst

$$J(\kappa)=\frac{1}{0.001+\sqrt{(\frac{x_f-x_0}{x_0})^2+(\frac{y_f-y_0}{y_0})^2+(\frac{z_f-z_0}{z_0})^2+(\frac{\dot{x}_f-\dot{x}_0}{\dot{x}_0})^2+(\frac{\dot{y}_f-\dot{y}_0}{\dot{y}_0})^2+(\frac{\dot{z}_f-\dot{z}_0}{\dot{z}_0})^2}} \tag{5}$$

A perfect individual (periodic motion) has a fitness value of 1000, while a percentage difference of 0.1% between the initial and the final state, brings down the fitness value to 500, a difference of 1% corresponds to a fitness value of 90, and so on.

III. Unperturbed Case

To test and tune our approach, we first consider the dynamic system describing the relative motion around a perfectly spherical Earth with uniform mass, and we compare our results with the semimajor axis matching condition that assures a perfectly periodic relative motion. We also perform comparisons with several other known approximate results coming from Taylor expansions of the original nonlinear system. These are described next.

A. HCW Case

Consider the Hill–-Clohessy–Wiltshire linear equations, valid for circular unperturbed reference orbits:

$$\begin{cases} \ddot{x}-2\omega\dot{y}-3\omega^2 x=0 \\ \ddot{y}+2\omega\dot{x}=0 \\ \ddot{z}+\omega^2 z=0 \end{cases} \tag{6}$$

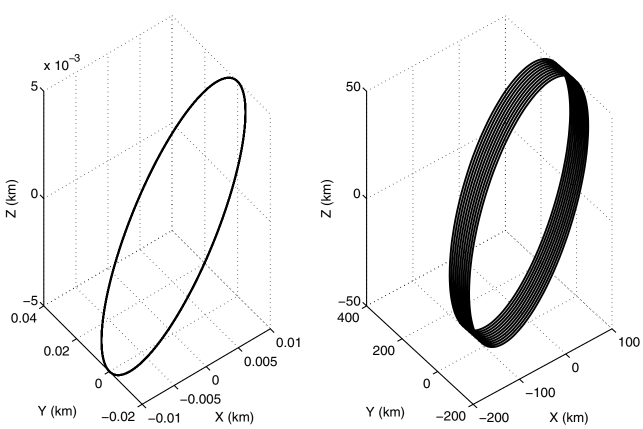

Fig. 1 Effects of the nonlinearities on the HCW condition for small (left) and large (right) formations.

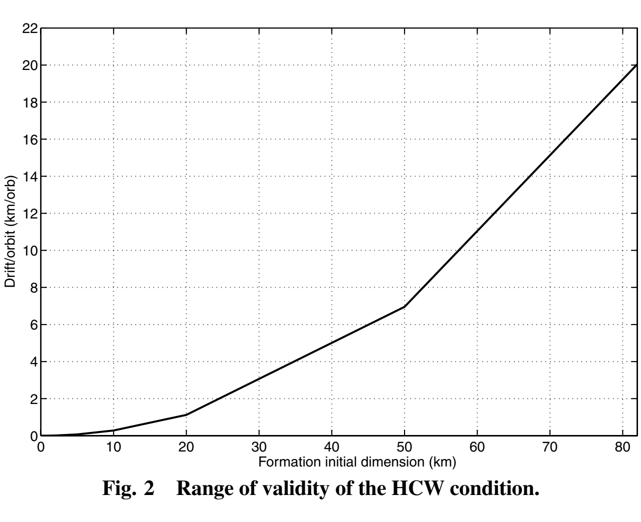

Fig. 2 Range of validity of the HCW condition.

The periodic motion condition, coming from the suppression of the secular term in the equations solution, is

$$\dot{y}_0=-2\omega x_0 \tag{7}$$

Trivially, as the formation dimension grows, the nonlinearities make this condition (which we refer to as the HCW condition) no longer valid, even when only Keplerian forces are considered. The well-known relative trajectories in Fig. 1 are obtained by propagating for 11 periods some initial conditions fulfilling the HCW condition using a nonlinear, nonperturbed model.

In Fig. 2 the drift per orbit obtained applying the condition in Eq. (7) is reported as a function of the formation dimension K. The initial relative position and velocities considered are $K[1,0,0.5,0,-2\omega,0]$ km. The drift per orbit is measured as the difference between the spacecraft relative distance at the initial time and after one period as obtained by propagating the dynamics with a nonlinear model.

B. Nonlinear Correction

Vaddi et al. [5] developed a model that takes into account the effects of nonlinearities, both for circular and for elliptical orbits. Following the same approach of the Taylor series expansion used to derive the HCW equations, but retaining also quadratic terms, one may obtain the following model:

$$\begin{cases} \ddot{x}-2\omega\dot{y}-3\omega^2 x=\varepsilon\left[\frac{y^2}{2}+\frac{z^2}{2}-x^2\right] \\ \ddot{y}+2\omega\dot{x}=\varepsilon xy \\ \ddot{z}+\omega^2 z=\varepsilon xz \end{cases} \tag{8}$$

where $\varepsilon=3\mu/a^4$. A condition for periodic relative orbits is then reached:

$$\begin{aligned} [x_0,y_0,z_0]&=\left[\frac{\rho}{2}\sin(\omega t+\alpha_0),\rho\cos(\omega t+\alpha_0),\rho\sin(\omega t+\alpha_0)\right] \\ [\dot{x}_0,\dot{y}_0,\dot{z}_0]&=\left[\frac{\omega\rho}{2}\cos(\omega t+\alpha_0),\dot{y}(0),\omega\rho\cos(\omega t+\alpha_0)\right] \end{aligned} \tag{9}$$

where ρ is the relative distance and α_0 the initial relative phase angle. The only variable influencing the secular growth of the relative motion is $\dot{y}$, which can be written as

$$\dot{y}(0)=\dot{y}_h(0)+\varepsilon\dot{y}_{\mathrm{cn}}(0) \tag{10}$$

where $\dot{y}_h$ is the term deriving from the HCW condition and $\dot{y}_{\mathrm{cn}}$ is the correction for the nonlinearity:

$$\dot{y}_{\mathrm{cn}}(0)=-\left(\frac{\rho^2}{48\omega}\right)[12+6\cos(2\alpha_0)] \tag{11}$$

C. Epicyclic Elements

A different analytical formulation is found in [4]. Here Kasdin and Koleman use a Hamiltonian approach to derive the equations of motion for an object relative to a circular or slightly elliptical reference orbit. By solving the Hamilton–Jacobi equation in terms of the epicyclic elements they are able to provide analytical approximations of the invariance condition. By means of this formalism, they derive bounded, periodic orbits in the presence of various perturbations, among them the nonlinearities. Here we only report the conditions found for the circular reference orbit case. Two expressions are given to compute a normalized $\bar{\dot{y}}_0$ [in Eq. (15) the

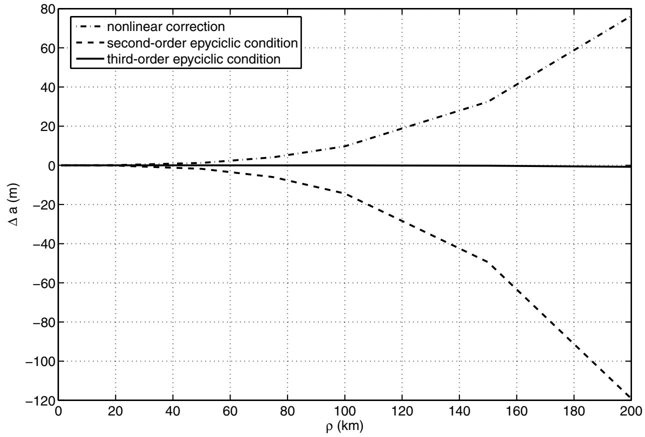

Fig. 3 Difference of semimajor axis vs initial dimensions.

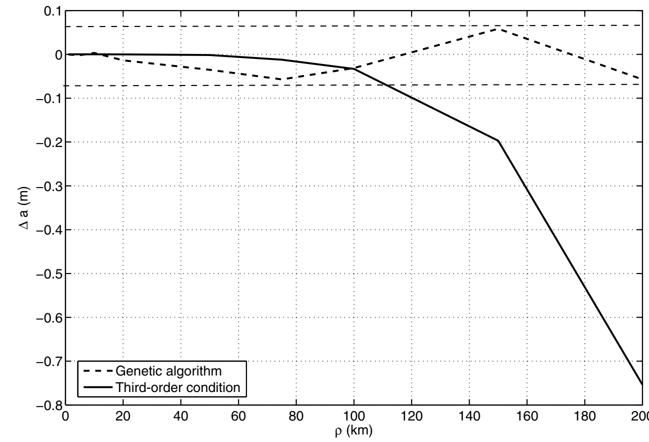

Fig. 4 Comparison between GA and third-order epicyclic condition.

bars stand for the distances being normalized by the reference orbit semimajor axis a, the time by the angular velocity ω, giving a-dimensional quantities]: 1) the first one considers second-order terms in the series expansion for the initial conditions, 2) the second considers also third-order terms:

$$a_3(0) = -\tfrac{5}{2}a_1^2(0) - \tfrac{1}{2}[a_2^2(0) - b_1^2(0) + b_2^2(0)] - 3a_1(0)b_3(0) - b_3^2(0) \tag{12}$$

$$a_3(0) = -\tfrac{5}{2}a_1^2(0) - \tfrac{1}{2}[a_2^2(0) - b_1^2(0) + b_2^2(0)] - 3a_1(0)b_3(0) - b_3^2(0) + -\tfrac{3}{2}[a_1^2(0)b_1(0) + a_2^2(0)b_1(0) + \tfrac{1}{2}b_1^3(0)] \tag{13}$$

In both cases:

$$a_1 = \sqrt{2\alpha_1}\cos\beta_1 \quad b_1 = \sqrt{2\alpha_1}\sin\beta_1 \quad a_2 = \sqrt{2\alpha_2}\cos\beta_2 \quad b_2 = \sqrt{2\alpha_2}\sin\beta_2 \quad a_3 = \alpha_3 \quad b_3 = \beta_3 \tag{14}$$

where α_i, β_i are the initial canonical momenta and coordinates, which can be written as functions of the initial conditions. For brevity we omit the subscript 0 in the following:

$$\alpha_1 = \frac{1}{2}[\bar{\dot{x}}^2 + (2\bar{\dot{y}} + 3\bar{x})^2] \quad \alpha_2 = \frac{1}{2}(\bar{\dot{z}}^2 + \bar{z}^2) \quad \alpha_3 = \bar{\dot{y}} + 2\bar{x}$$
$$\beta_1 = -\tan^{-1}\left(\frac{3\bar{x} + 2\bar{\dot{y}}}{\bar{\dot{x}}}\right) \quad \beta_2 = -\tan^{-1}\left(\frac{\bar{z}}{\bar{\dot{z}}}\right) \quad \beta_3 = -2\bar{\dot{x}} + \bar{y} \tag{15}$$

Substituting Eq. (15) in Eq. (14), using the conditions in Eq. (12) or in Eq. (13) (according to the order of the chosen approximation), and solving for $\dot{y}$, gives the initial $\dot{y}$ for bounded orbits.

The difference between the semimajor axes of the spacecraft in the formation is a good index of how near the approximation of the analytical conditions is to the physical one (i.e., semimajor axis matching); a measure of the drift per orbit can be given as [9]

$$\text{drift}/\text{orbit} = -3\pi\Delta a \tag{16}$$

The difference Δa resulting by using condition (11) and (12) or (13) can be plotted for various formation dimensions as shown by Fig. 3. The third-order epicyclic conditions are a very good approximation of the period-matching condition, and indeed the use of a numerical approach is not necessary in this case. We rather used these analytical results and the period-matching condition to test the performances of the numerical technique and to tune the optimizer parameters.

The final comparison between the best analytical (third-order epicyclic) and numerical (genetic algorithm without final local refinement) solutions is showed in Fig. 4.

Figure 4 shows the main difference between the analytical and numerical approach: the Δa resulting from the genetic algorithm simulations oscillates because of the stochastic nature of the optimizer, while the Δa resulting from the application of the third-order conditions grows with the formation dimensions. Similar results can be obtained for elliptical unperturbed reference orbits.

IV. J_2 Perturbed Case

Let us study the solutions of Eq. (1) in the case where **f** describes the relative motion between two satellites orbiting around an oblate Earth. The objective function J is again given by Eq. (5). As already mentioned, this corresponds to minimizing the relative orbital drift. Some previous work has been done to determine the possibility of invariant relative satellite motion when J_2 is considered as a perturbing term. In particular, the paper by Schaub and Alfriend [7] introduces the so-called J_2 invariant relative orbits. In their work, mean orbital elements are used and the secular drifts of the longitude of the ascending node and of the sum of the argument of perigee and mean anomaly are set to be equal between two neighboring orbits. By having both orbits drifting at equal angular rates on the average, they will not separate over time due to the J_2 influence. Two first-order conditions are presented in [7]:

$$\delta a = 2Da_0\delta\eta \qquad \delta e = \frac{(1-e^2)\tan(i)}{4e}\delta i \tag{17}$$

where

$$\delta\eta = -\frac{\eta_0}{4}\tan(i_0)\delta i \qquad \eta = \sqrt{1-e^2} \tag{18}$$

and D is a parameter depending on i, a, and η. The combination of Eqs. (17) and (18) provides the two necessary conditions on the mean orbital element differences yielding a J_2-invariant relative orbit. When designing a relative orbit using the mean orbital element differences, δi, δe, or δa is chosen, the remaining two element differences are then found through the two constraints in Eq. (18). The remaining element differences $\delta\Omega$, $\delta\varpi$, and δM can then be chosen at will without affecting the J_2 invariance. Even though called J_2 invariant orbits, these two conditions are only valid in a first-order approximation. When using these conditions the relative orbit is still exhibiting a relative drift, as Fig. 5 shows for an almost circular 35 deg inclined reference orbit. Propagation is again performed via a nonlinear model including J_2 as a perturbation.

The conditions in Eqs. (17) and (18) represent two elegant relations defining relative orbits with a small drift per orbit. We use our numerical approach based on the solution of the global optimization problem stated in Eq. (1) to check to what extent the residual drift obtained with this analytical approach is an artifact of the use of mean elements. Repeating the calculation for the entire range of inclinations, the results vary sensibly, disclosing a previously unknown feature of invariant relative motion. In Fig. 6, we report the final fitness function reached for different inclinations ranging from 0 to 180 deg. The other orbital parameters of the Chief

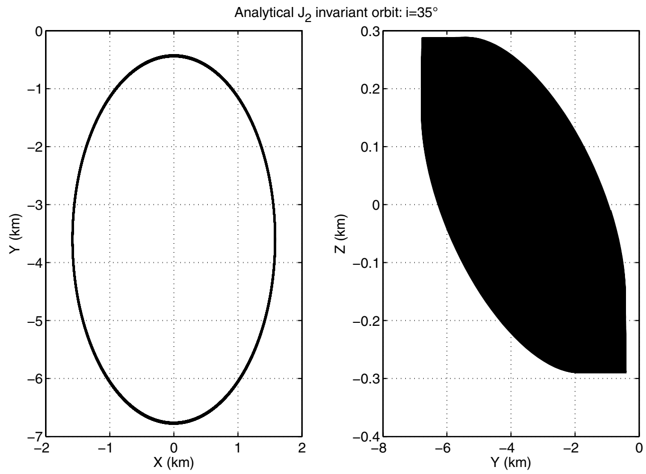

Fig. 5 100 relative orbits for a J_2 perturbed case at 35 deg inclination (using the J_2 invariance condition).

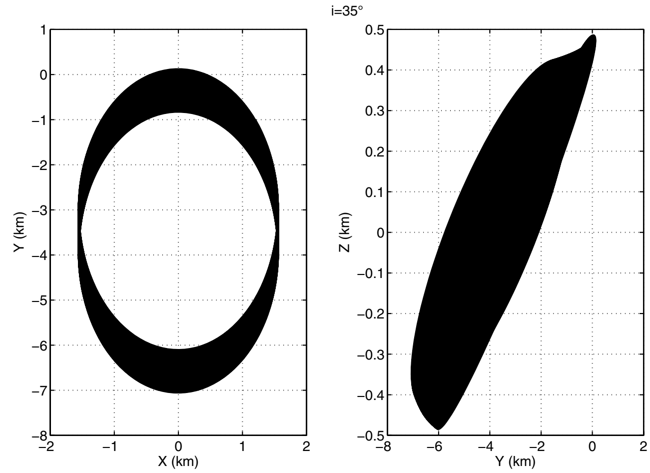

Fig. 7 100 relative orbits for a J_2 perturbed case at 35 deg inclination (best individual).

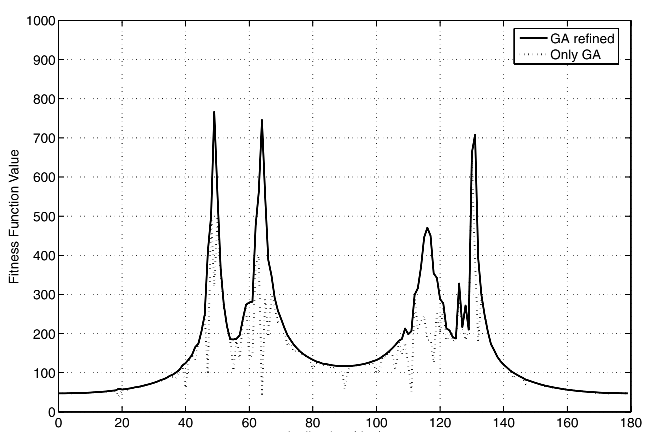

Fig. 6 Best individual fitness value vs inclination of reference orbit.

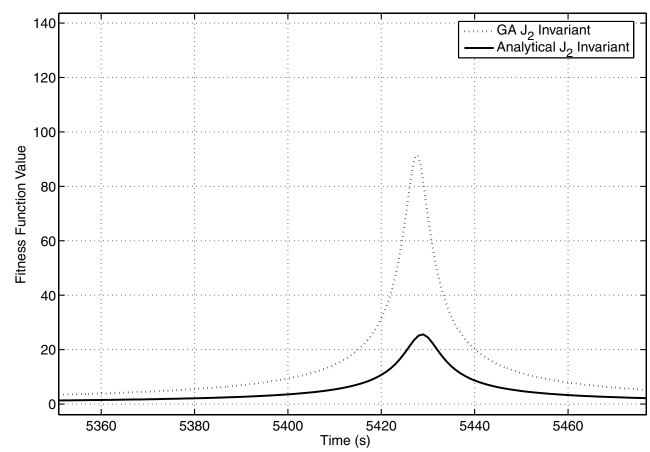

Fig. 8 Comparison with a Schaub invariant orbit condition at 35 deg.

satellite used for this simulation were $a = 6678$ km, $e = 0.00118$, $\varpi = 90$ deg, $\Omega = 270$ deg, and $\theta = 0$ deg. The size of the relative orbit is regulated by means of the constant K in Eqs. (3) and (4): in this simulation this is set to 1. In the figure, we report both the output from the genetic algorithm and the final solution obtained refining the solution with a local optimization.

For all inclination the minimal drift is not zero, with two remarkable exceptions: 49 and 63.4 deg, and their symmetric counterparts (with respect to 90 deg), that is, 131 and 116.6 deg. In the following we will refer to the 49 and 131 deg inclinations as "special inclinations," keeping the term "critical inclinations" for the 63.4, 116.6 deg case. The heuristic of the genetic algorithms is definitely not responsible for these peaks, as it turns out by actually propagating the resulting best individuals. At a generic inclination, say 35 deg, the best individual returned by the optimization results in a relative motion that is not periodic is visualized in Fig. 7. The small residual drift is comparable to the one that results using the Schaub J_2-invariant orbit condition. To confirm this last statement, we have plotted in Fig. 8 the value of the objective function given by Eq. (5) during one complete orbit in the case of the Schaub J_2-invariant orbit and in the case of the condition returned by our genetic algorithm.

At critical inclinations (63.4, 116.6 deg) the relative motion turns out to be perfectly periodical (see Fig. 9). The corresponding optimal decision vector is

$$\kappa = [0.602\ \text{km}, 0.848\ \text{km}, 0.06\ \text{km}, -5.32 \times 10^{-3}\ \text{km/s}, \\ -1.402 \times 10^{-3}\ \text{km/s}, -9.339 \times 10^{-3}\ \text{km/s}, 5.452.7\ \text{s}]$$

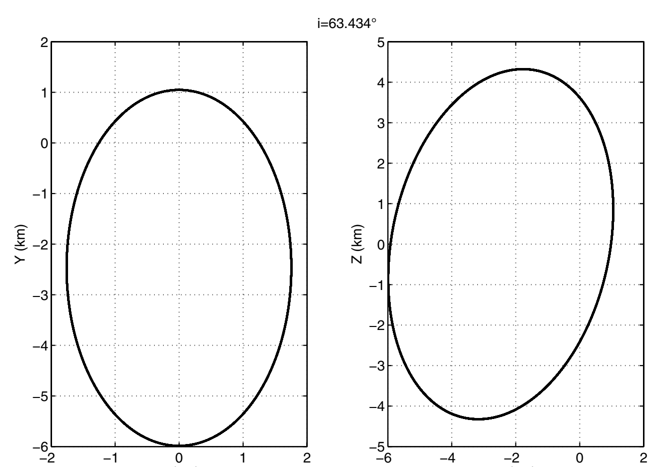

Fig. 9 100 relative orbits for a J_2 perturbed case at 63.4 deg inclination (best individual).

Note that in the J_2 perturbed case, the condition is no longer of period matching as a difference in all the six orbital elements is kept, as reported in Table 2.

The possibility of obtaining a perfectly periodical relative motion at these inclinations is probably linked to the cancellation of the secular drift of the perigee argument, which causes the variation of all parameters to happen with the same main frequency.

Table 2 Comparing orbital elements for the J_2 case at the critical inclination

Orbital element	Chief	Deputy	Difference
a	6678 km	6677.7091 km	−0.291 km
e	0.00118	0.01573	0.01455
i	63.435 deg	63.391 deg	−0.044 deg
ϖ	90 deg	50.126 deg	−39.87
Ω	270 deg	−89.123 deg	0.877 deg
θ	0 deg	40.333 deg	40.333 deg
$\varpi + \theta$	90 deg	90.46 deg	0.46 deg

At the two special inclinations (49, 131 deg), the relative satellite motion resulting from the best individual has only a very small drift, as shown in Fig. 10. The corresponding decision vector is

$$\kappa^{\bullet} = [1.652 \text{ km}, 2.692 \text{ km}, 2.546 \text{ km}, 1.689 \times 10^{-3} \text{ km/s}, \\ -3.839 \times 10^{-3} \text{ km/s}, -4.120 \times 10^{-3} \text{ km/s}, 5.436.77 \text{ s}]$$

The residual drift does not allow us to conclude that the motion is perfectly periodical at these inclinations. In fact, we were unable to find a fitness value of 1000 (meaning perfect periodicity) at all the inclinations. A more detailed plot of the objective function achieved around the special inclination is shown in Fig. 11. The numeric noise that can be observed in the graph is a consequence of the numerical optimization process, amplified by the definition of the objective function given by Eq. (5). At higher values of the fitness very small differences in the residual drift cause significant differences in the objective function value. We show in Fig. 12 a plot of the period of the found minimum drift orbits. This is clearly quite different from the Keplerian period confirming the importance of having the optimizer to choose it.

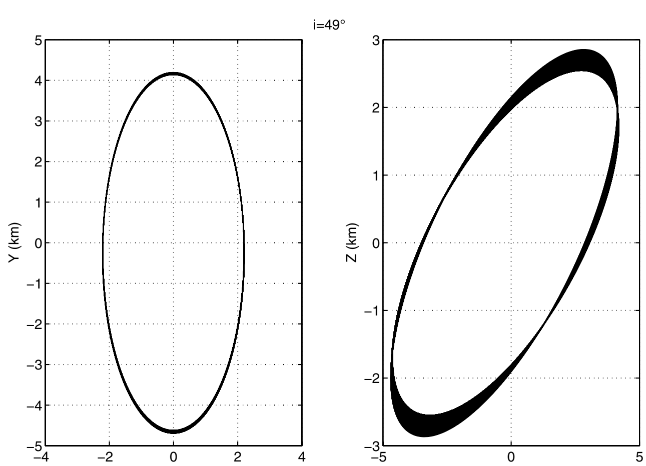

Fig. 10 100 relative orbits for a J_2 perturbed case at 49 deg inclination (best individual).

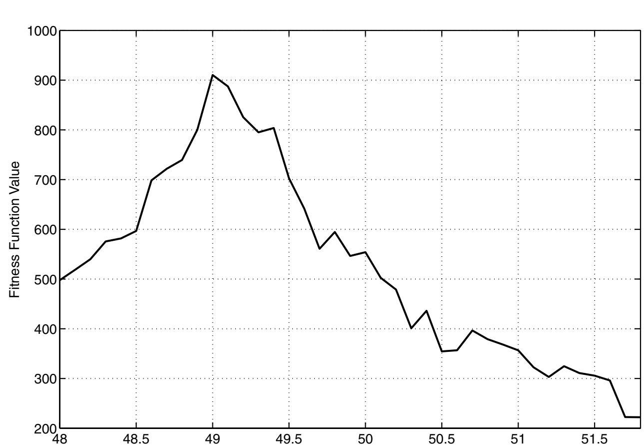

Fig. 11 Details around a *special* inclination.

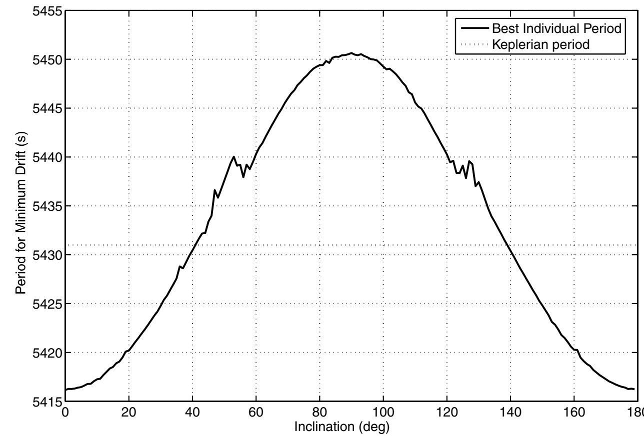

Fig. 12 Best individual period (s).

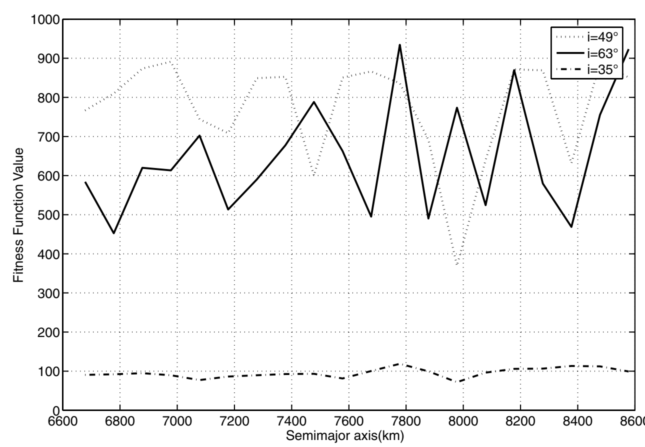

Fig. 13 Best individual fitness vs increased values for the semi-axis.

The existence of the two special inclinations where the relative motion between satellites can be periodical is evident from the data presented in Fig. 6 and clearly calls for some explanation. A number of elegant and interesting exact results have already been established in the years for the periodicity of absolute satellite motion. Kyner [10] deals with the periodicity of a J_2 perturbed motion. Hughes [11] discusses the occurrence and the uniqueness of the critical inclinations. Mortari [12] developed an entire new theory to deal with the periodicity of satellite constellations with respect to different reference frames. Unfortunately, the case of relative satellite motion has been studied much less. We have already commented on the approximated minimal drift conditions available in the literature, but to the best knowledge of these authors, there are no exact results on the subject. Recent studies [13,14] try to approach the phenomenon, though not giving a definite answer.

One may argue that the existence and the value of the two special inclinations may be linked to the parameters of the Chief orbit or to the size of the formation (i.e., the relative orbit typical dimension). By increasing the semimajor axis of the Chief orbit the fitness values reached by formations at both $i = 49$ deg and $i = 63$ deg are not influenced as shown in Fig. 13.

The same results apply by changing the value K, related to the formation size as shown if Fig. 14. The validity of the special inclinations is not affected by the size of the formation or by the semimajor axis of the Chief orbit: they exist for a wide range of formations around circular orbits. For the eccentricity the results are quite different.

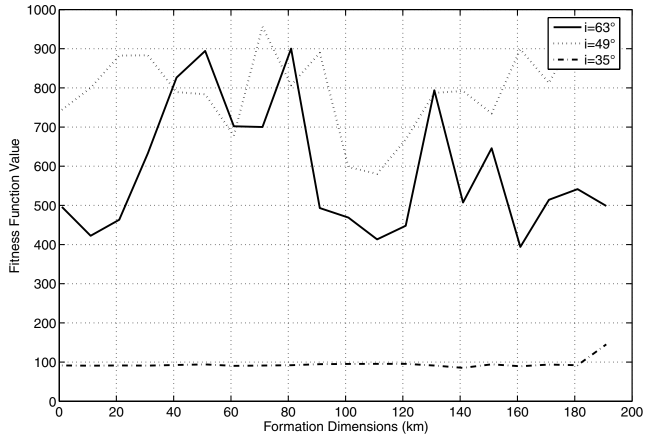

Fig. 14 Best individual fitness vs formation size.

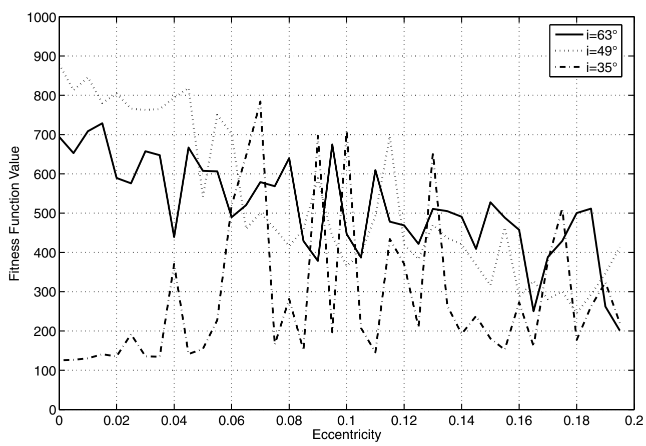

Fig. 15 Best individual fitness vs eccentricity.

Let us plot the result obtained by increasing the eccentricity value of the Chief orbit. The critical and the special inclinations are again compared to a general case; the results are reported in Fig. 15. To allow the study of high eccentricity, the semimajor axis of the reference orbit has been raised to 8000 km. The best individual fitness value for the special and the critical inclinations is quite high for the range of eccentricity analyzed, but surprisingly also at the generic inclination, the fitness may present remarkable values. Therefore, in the case of eccentric orbits, the drift in the relative motion can be small also outside of the critical and the special inclinations.

V. Conclusions

The possibility to obtain natural periodic motion of formation flying satellites has been investigated through the use of a numerical global optimization technique such as genetic algorithms, refined by a constrained nonlinear optimization. After validating the approach on the well-known unperturbed test case, the attention has been focused on perturbations. Although some results obtained are trivial and expected, some others are quite surprising and interesting. In particular, the possibility to have a periodic motion is denied, as shown later on, and also when a conservative, symmetric perturbation like J_2 is considered. We find four remarkable exceptions: when the formation reference orbit (circular or elliptical) has an inclination of 63.4 or 116.6 deg (which are the classical critical inclinations) and of 49 or 131 deg (which we define as special inclinations). An extensive simulation campaign is performed to test the relative motion features at these special inclinations. While the physical reasons of this behavior are still under study, a simple conclusion can be drawn. If two satellites have to remain in close formation, the choice of the inclination of the reference orbit is of fundamental importance, and it results in a smaller control effort to keep the satellites in a bounded formation.

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